

# Internet Appendix For Cojumps and Asset Allocation in International Equity Markets

Mohamed Arouri<sup>a</sup>, Oussama M'saddek<sup>b</sup>, Duc Khuong Nguyen<sup>c,\*</sup>, Kuntara Pukthuanthong<sup>d</sup>

<sup>a</sup>*University of Nice Sophia Antipolis, Nice, France*

<sup>b</sup>*University of Auvergne, Clermont-Ferrand, France*

<sup>c</sup>*IPAG Business School, Paris, France*

<sup>d</sup>*University of Missouri, Columbia, United States*

---

This Internet Appendix reports the details of the intraday volatility pattern corrector (Appendix A), the details of the mean-CVaR optimization problem (Appendix B) and the detailed resolution of the CRRA utility maximization problem (Appendix C).

---

\*Corresponding author: 184 Boulevard Saint-Germain, 75006 Paris, France. Phone: +33 1 5363 3600. Fax: +33 1 4544 4046.

## Appendix A. Intraday volatility pattern

It is widely documented (Wood et al. (1985) and Harris (1986)) that intraday returns show a systematic seasonality over the trading day, also called the U-shaped pattern. The intraday volatility is particularly higher at the open and the close of the trading than the rest of the day.

To minimize the effects of intraday volatility on our jump detection test, we modify our procedure by rescaling intraday returns with a volatility jump robust corrector introduced by Bollerslev et al. (2008). The  $k^{th}$  rescaled intraday return of day  $t$  is defined by:

$$\hat{r}_{t,k} = \frac{r_{t,k}}{\varsigma_k}$$

where:

$$\begin{aligned} \varsigma_1^2 &= \frac{M \sum_{t=1}^T |r_{t,1}| |r_{t,2}|}{\sum_{t=1}^T |r_{t,1}| |r_{t,2}| + \sum_{t=1}^T \sum_{l=2}^{M-1} |r_{t,l-1}|^{\frac{1}{2}} |r_{t,l}| |r_{t,l+1}|^{\frac{1}{2}} + \sum_{t=1}^T |r_{t,M-1}| |r_{t,M}|} \\ \varsigma_k^2 &= \frac{M \sum_{t=1}^T |r_{t,k-1}|^{\frac{1}{2}} |r_{t,k}| |r_{t,k+1}|^{\frac{1}{2}}}{\sum_{t=1}^T |r_{t,1}| |r_{t,2}| + \sum_{t=1}^T \sum_{l=2}^{M-1} |r_{t,l-1}|^{\frac{1}{2}} |r_{t,l}| |r_{t,l+1}|^{\frac{1}{2}} + \sum_{t=1}^T |r_{t,M-1}| |r_{t,M}|}, \quad k = 2, \dots, M-1 \\ \varsigma_M^2 &= \frac{M \sum_{t=1}^T |r_{t,M-1}| |r_{t,M}|}{\sum_{t=1}^T |r_{t,1}| |r_{t,2}| + \sum_{t=1}^T \sum_{l=2}^{M-1} |r_{t,l-1}|^{\frac{1}{2}} |r_{t,l}| |r_{t,l+1}|^{\frac{1}{2}} + \sum_{t=1}^T |r_{t,M-1}| |r_{t,M}|} \end{aligned}$$

$T$  is the total number of days considered in the study and  $M$  is the number of observations in a day.

## Appendix B. Mean-CVaR optimization problem

The mean-CVaR optimization approach initially developed by Rockafellar and Uryasev (2000) can be described as follows. We first define the loss function of a portfolio composed of  $n$  assets given the vector of weights  $w$  and the random vector of asset returns  $r$  such as

$$f(w, r) = - \sum_{i=1}^n w_i r_i = -w' r$$

The probability of  $f(w, r)$  not exceeding a threshold  $\alpha$  is given by:

$$\Psi(w, \alpha) = \int_{f(w, r) \leq \alpha} p(r) dr$$

where  $p(r)$  is the density function of the vector of returns.  $\Psi$  is a function of  $\alpha$  for a fixed vector of weights  $w$  and represents the cumulative distribution function for the loss associated with the vector of weights  $w$ .

The values of the VaR and the CVaR of the loss function associated with  $w$  and a confidence level  $\beta$ ,  $\alpha_\beta(w)$  and  $\phi_\beta(w)$ , can be then determined as:

$$\begin{aligned} \alpha_\beta(w) &= \min(\alpha \in R : \Psi(w, \alpha) \geq \beta) \\ \phi_\beta(w) &= (1 - \beta)^{-1} \int_{f(w, r) \geq \alpha_\beta(w)} f(w, r) p(r) dr \end{aligned}$$

Following Rockafellar and Uryasev (2000), we provide the expression  $\phi_\beta(w)$  using the function  $F_\beta$  defined as follows:

$$F_\beta(w, \alpha) = \alpha + (1 - \beta)^{-1} \int [f(w, r) - \alpha]^+ p(r) dr$$

where  $[x]^+ = \max(x; 0)$ . Rockafellar and Uryasev (2000) show that  $F_\beta(w, \alpha)$  is convex and continuously differentiable as a function of  $\alpha$ . It is also related to the CVaR of the loss function through  $\phi_\beta(w) = \min_{\alpha \in R} (F_\beta(w, \alpha))$ . Moreover, the authors prove that minimizing  $\phi_\beta(w)$  over all  $w \in R^n$  is equivalent to minimizing  $F_\beta(w, \alpha)$  over all  $(w, \alpha) \in R^n \times R$ , that is  $\min_{w \in R^n} \phi_\beta(w) = \min_{(w, \alpha) \in R^n \times R} F_\beta(w, \alpha)$ . The expression of  $F_\beta$  can be simplified by generating a random collection of the vector of returns  $(r^{(1)}, r^{(2)}, \dots, r^{(q)})$  and approximated with  $\tilde{F}_\beta$  as follows:

$$\tilde{F}_\beta(w, \alpha) = \alpha + \frac{1}{q(1 - \beta)} \sum_{i=1}^q [f(w, r^{(i)}) - \alpha]^+$$

Replacing the loss function by its expression gives:

$$\tilde{F}_\beta(w, \alpha) = \alpha + \frac{1}{q(1 - \beta)} \sum_{i=1}^q [-w' r^{(i)} - \alpha]^+$$

By introducing the auxiliary variable  $u_i$ , the minimizing of  $\tilde{F}_\beta$  is equivalent to the linear equation:

$$\begin{aligned} &\alpha + \frac{1}{q(1 - \beta)} \sum_{i=1}^q u_i \\ &\text{subject to: } u_i \geq 0, u_i + w' r^{(i)} + \alpha \geq 0 \quad \text{for } i = 1, \dots, q \end{aligned}$$

If we add the budget and the expected target return constraints, the portfolio mean-CVaR optimization problem is given by:

$$\begin{aligned} \min_{\alpha, w} \quad & \alpha + \frac{1}{q(1-\beta)} \sum_{i=1}^q u_i \\ \text{subject to} \quad & \begin{cases} e'w = 1, u_i \geq 0 \\ \mu'w = \bar{\mu} \\ u_i + w'r^{(i)} + \alpha \geq 0, i = 1, \dots, q \end{cases} \end{aligned}$$

### Appendix C. Expected CRRA utility maximization

The investor's problem at time  $t$  is given by :

$$\begin{aligned} V(t, W_t) = \max_w E[U(W_T)] \\ \text{subject to } e'w + w_0 = 1 \end{aligned}$$

where  $w = (w_1, w_2, \dots, w_n)'$  is the vector of portfolio weights of  $n$  risky assets.  $w_0$  is the weight of the riskless asset.  $e = (1, 1, \dots, 1)'$  denotes the vector of ones and  $U$  is the CRRA utility function with a constant relative risk aversion coefficient  $\gamma$ . The dynamics of the wealth  $W_t$  is as follows:

$$\frac{dW_t}{W_t} = (w'R + r)dt + w'(\sigma \cdot dZ_t) + \sum_{x \in (up, down)} w' J^{sys, x} dQ^{sys, x} + \sum_{x \in (up, down)} w'(J^{id, x} \cdot dQ^{id, x})$$

where  $R = (\mu_1 - r, \mu_2 - r, \dots, \mu_n - r)'$  is the diffusive excess-returns vector.  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)'$  the vector of diffusive volatilities.  $Z_t = (Z_{t,1}, Z_{t,1}, \dots, Z_{t,n})'$  the vector of Brownian motions.  $J^{id, x} = (J_1^{id, x}, J_2^{id, x}, \dots, J_n^{id, x})'$  is the vector of idiosyncratic (up or down) jump amplitudes whereas  $J^{sys, x} = (J_1^{sys, x}, J_2^{sys, x}, \dots, J_n^{sys, x})'$  denotes the vector of systematic (up or down) jump amplitudes.

$Q^{id, x} = (Q_1^{id, x}, Q_2^{id, x}, \dots, Q_n^{id, x})'$  is the vector of idiosyncratic jump Poisson processes. The  $\cdot$  operator denotes the element- by-element multiplication of two equally sized vectors.

By applying Ito's lemma to  $V(t, W_t)$  and using stochastic dynamic programming techniques, we obtain the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} 0 = \max_w \left\{ \frac{\partial V(t, W_t)}{\partial t} + \frac{\partial V(t, W_t)}{\partial W_t} W_t (w'R + r) + \frac{1}{2} \frac{\partial^2 V(t, W_t)}{\partial W_t^2} W_t^2 w' \Sigma w \right. \\ \left. + \sum_{x \in (up, down)} \lambda^{sys, x} E[V(t, W_t + W_t w' J^{sys, x}) - V(t, W_t)] \right. \\ \left. + \sum_i \sum_{x \in (up, down)} \lambda_i^{id, x} E[V(t, W_t + W_t w_i J_i^{id, x}) - V(t, W_t)] \right\} \end{aligned}$$

where  $\Sigma = (\rho_{i,j}\sigma_i\sigma_j)_{1 \leq i,j \leq n}$  is the covariance matrix of the diffusive returns.

We guess that the function  $V(t, W_t)$  is of the following form :

$$V(t, W_t) = A(t) \frac{W_t^{1-\gamma}}{1-\gamma}$$

Using this guess, the Hamilton-Jacobi-Bellman equation writes :

$$\begin{aligned} 0 = & \max_w \left\{ \frac{1}{A(t)} \frac{dA(t)}{dt} + (1-\gamma)(w'R + r) - \frac{(1-\gamma)\gamma}{2} w' \Sigma w \right. \\ & + \sum_{x \in (up, down)} \lambda^{sys,x} E[(1 + w' J^{sys,x})^{1-\gamma} - 1] \\ & \left. + \sum_i \sum_{x \in (up, down)} \lambda_i^{id,x} E[(1 + w_i J_i^{id,x})^{1-\gamma} - 1] \right\} \end{aligned}$$

Differentiating the above equation with respect to  $w$ , we obtain the following system of non linear equation :

$$\mathbf{0} = R - \gamma \Sigma w + \sum_{x \in (up, down)} \lambda^{sys,x} E[J^{sys,x} (1 + w' J^{sys,x})^{-\gamma}] + \sum_{x \in (up, down)} \lambda^{id,x} E[J^{id,x} (e + w \cdot J^{id,x})^{-\gamma}]$$

where  $\mathbf{0} = (0, 0, \dots, 0)'$  is the vector of zeros.  $\lambda^{id,x} = (\lambda_1^{id,x}, \lambda_2^{id,x}, \dots, \lambda_n^{id,x})'$  is the vector of idiosyncratic (up or down) jump intensities.

By evaluating the Hamilton-Jacobi-Bellman equation at the optimal portfolio weights  $w^*$ , we obtain :

$$\begin{aligned} \frac{1}{A(t)} \frac{dA(t)}{dt} &= -\kappa \\ \kappa &= (1-\gamma)(w^{*'} R + r) - \frac{(1-\gamma)\gamma}{2} w^{*'} \Sigma w^* + \sum_{x \in (up, down)} \lambda^{sys,x} E[(1 + w^{*'} J^{sys,x})^{1-\gamma} - 1] \\ &+ \sum_i \sum_{x \in (up, down)} \lambda_i^{id,x} E[(1 + w_i^* J_i^{id,x})^{1-\gamma} - 1] \end{aligned}$$

By integrating the differential equation, we get :

$$A(t) = e^{\kappa(T-t)}, A(T) = 1$$

## References

Bollerslev, T., Law, T. H., Tauchen, G., 2008. Risk, jumps, and diversification. *Journal of Econometrics* 144 (1), 234-256.

Harris, L., 1986. A transaction data study of weekly and intradaily patterns in stock returns. *Journal of Financial Economics* 16, 99-117.

Rockafellar, R. T. and Uryasev, S., 2000. Optimization of Conditional Value-at-Risk. *Journal of Risk* 2, 21-41.

Wood, R. A., McInish, A., Thomas H., and Ord, K., 1985. An investigation of transaction data for NYSE stocks. *Journal of Finance* 40, 723-741.